☐ Mic check ☐ Change of plans (Donaldson soon) ☐ Feedback
☐ Record ☐ Exam announcement (July 17-19, Sep 11-13)

$\chi_4(\hat{\beta}) \leq n - wr(\beta)$

Last Time: braids, SBI, SQP

1 Khovanov Homology (Kh) - What is it?

It's a functor:

$\xrightarrow{Kh}$

$$\begin{array}{c} Kh(L_1) \\ \uparrow Kh(S) \\ Kh(L_0) \end{array}$$

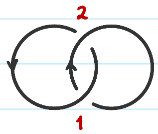
Think concordance but can have genus

More rigorously: $S \subseteq \mathbb{R}^3 \times [0,1]$ is any **smooth**, compact, not-necessarily oriented, properly-embedded surface with (potentially empty) boundary links $L_0 \sqcup L_1 \subseteq \mathbb{R} \times \{0,1\}$

Blank space

② Khovanov Homology of Links

To ort.^{d} link L with diagram D we will assign a chain cx $\mathcal{C}(D)$

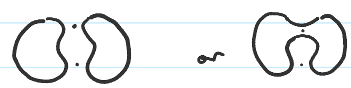


$n = 2, \quad n_+ = 2, \quad n_- = 0$

Ⓐ Enumerate crossings

Defn A *labeled smoothing* of D is a pair $\alpha = (\sigma, \ell)$

- a *smoothing* σ obtained by replacing all X'ings $\times$ of D with either a 0-smoothing $)\cdot($ or a 1-smoothing $\asymp$

  e.g. 

  and can be encoded as a binary sequence $\sigma = (\sigma_1, \dots, \sigma_n)$

- a *label* ℓ consisting of elements v_+ or v_- for every component in σ

  e.g. 

Ⓑ Chain Groups: $\mathcal{C}(D) = \langle \text{labeled smoothings} \rangle_{\mathbb{Z}}$

Defn $\mathcal{C}(D)$ is bigraded by:

- *homological grading* $h(\alpha) = \#\,\text{1-smoothings} - n_-$
- *quantum grading* $q(\alpha) = (\# v_+) - (\# v_-) + h(\alpha) + \underbrace{n_+ - n_-}_{w(D)}$

these just ensure that when we take homology, it is independent of the chosen diagram

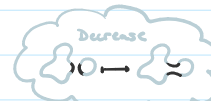
so $\mathcal{C}(D) = \bigoplus_{h,q} \mathcal{C}^{h,q}(D)$

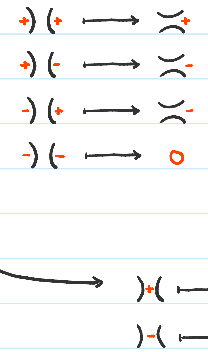
DIFFERENTIAL

Ⓒ Define $d^h : \mathcal{C}^{h,q}(D) \longrightarrow \mathcal{C}^{h+1,q}(D)$ on each labeled smoothing $\alpha = (\sigma, \ell)$ by:

for each $\sigma_i = 0$, create a new labeled smoothing α_i by

1. setting $\sigma_i = 1$
2. changing label by the following rule
   a. if # of components decreases, use map
   b. if # of components increases, use map
3. setting $\xi_i = \sum_{j<i} \sigma_j$ (ensures $d \circ d = 0$)

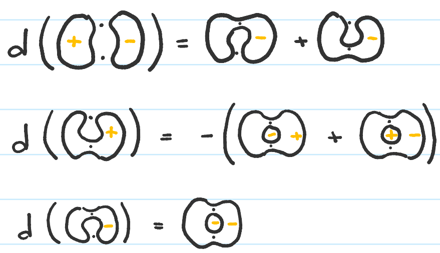




then set $d(\alpha) = \sum (-1)^{\xi_i} \alpha_i$

Extend linearly across $\mathcal{C}(D)$

e.g.



FACT d^h is a codifferential wrt h

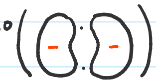
Defn The Khovanov chain cx is the pair $(\mathcal{C}(D), d)$ and has associated homology $Kh(D)$.

Thm (Kh '00) $Kh(D)$ is independent of D and enumeration, up to iso
$Kh(L) :=$ iso class of $Kh(D)$ — Reid. moves don't change homology

Remark $\mathcal{C}^{0,0}(\varnothing) = \mathbb{Z}$, $Kh^{0,0}(\varnothing) = \mathbb{Z}$
categorifies Jones polynomial
can have torsion (HW)
HW has some nice properties

2 EXAMPLE $Kh(\text{Hopf}) = \mathbb{Z}^4$ (HW)

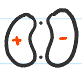
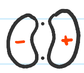
Can write d^h as a matrix and do linear algebra
Can also see explicit generators:

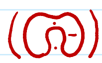
$d^{0,0}\Big($  $\Big) = 0 + 0 \Rightarrow \beta$ is a cycle

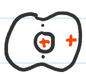
β (under the diagram, with "=")

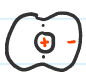
β is not a bndry because $\mathcal{C}^{-1,0}(D)$ is trivial

Other cycles:

 $-$  $\in \mathcal{C}^{0,2}$

~~ $\in \mathcal{C}^{2,2}$~~ $= \partial\Big($  $\Big)$

 $\in \mathcal{C}^{2,4}$

 $\in \mathcal{C}^{2,6}$

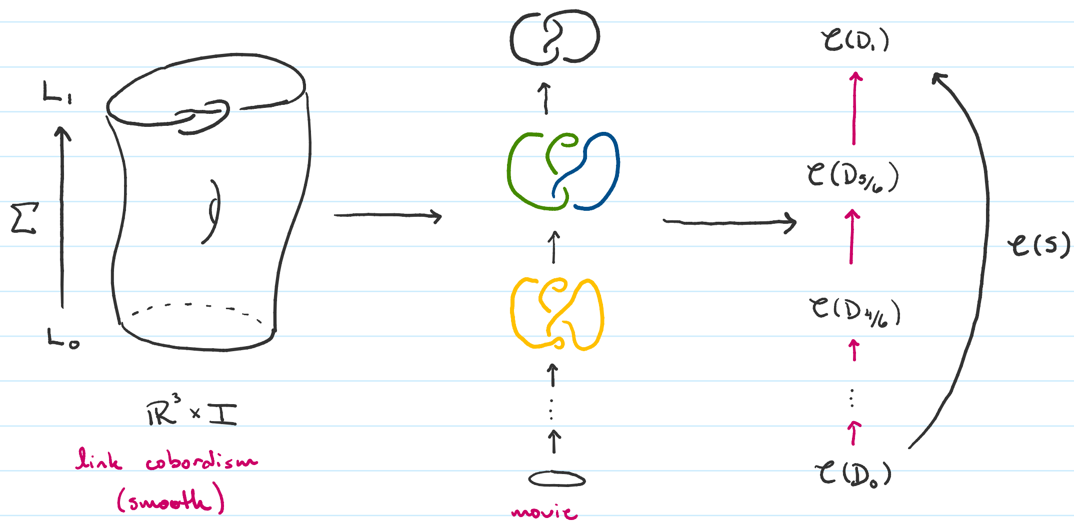
 $\in \mathcal{C}^{2,6}$

homologous up to sign

$Kh(\text{Hopf}) = \mathbb{Z}^4$

	0	1	2
6			$\mathbb{Z}$
4			$\mathbb{Z}$
2	$\mathbb{Z}$		
0	$\mathbb{Z}$		

3 Khovanov Homology of Surfaces



Given a link cob. $S : L_0 \to L_1$ and a movie $D_0, \dots, D_1$

a. consider chain cx's $\mathcal{C}(D_i)$ 

b. define maps $\mathcal{C}(D_i) \to \mathcal{C}(D_{i+1})$ induced by isotopy, Reidemeister moves, Morse moves

c. compose to get $\mathcal{C}(S)$

d. take homology to get $Kh(S)$

<u>FACTS</u> $Kh(S)$ is invariant up to iso rel ∂

$Kh(S)$ is $(0, \chi(S))$ bigraded, ie. $Kh(S) : Kh^{h,q}(L_0) \to Kh^{h,q+\chi(S)}(L_1)$